INTERSECTING LEGENDRIANS AND BLOW-UPS

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ABSTRACT. The purpose of this note is to describe the relationship between two classes of Legendre distributions. These two classes are distributions associated to an intersecting pair of Legendre submanifolds, introduced in [2] by analogy with intersecting Lagrangian distributions of Melrose and Uhlmann [8], and Legendre distributions associated to a fibred scattering structure introduced in [3]. We prove a general result, and also give an example in two dimensions, which shows explicitly the relation between the two spaces in a simple setting.

1. Introduction

The purpose of the present note is to clarify the relationship between two classes of Legendre distributions. The first class, that of intersecting Legendrians associated to a pair of Legendre manifolds which intersect cleanly, was defined by one of us in [2] as an analog of the notion of intersecting Lagrangian distributions [8] of Melrose and Uhlmann. Just as the latter played an important role in the study of real principal type operators, the former proved useful in geometric scattering theory, both in describing the structure of the boundary value of the resolvent (of a scattering Laplacian) at the real axis [3] and in the study of three-body scattering [2].

The second class of Legendre distributions, that of Legendrians associated to a fibred-scattering structure, was defined in [3]. This extends the notion of Legendre distributions to manifolds with corners that are equipped with certain boundary fibrations, and it was used to analyze the structure of the resolvent of scattering differential operators near the the corners of the b-double space.

Our result is then that, given appropriate geometry and symbolic orders, the class of intersecting Legendre distributions is a proper subset of Legendre distributions associated to a fibred scattering structure — see Theorem 3.2 in Section 3 for the precise statement. The proper inclusion corresponds to, roughly speaking, half of the possible terms in a Taylor series expansion of a general fibred-scattering Legendre distribution *not* being present in an intersecting Legendre distribution.

Lagrangian distributions on a manifold without boundary, X_0, are distributions on X_0 with very special singularities (in the sense of lack of smoothness) associated to Lagrangian submanifolds of the cotangent bundle of X. The simplest examples are conormal distributions to an embedded submanifold $Z \subset X_0$; these are distributions whose regularity is maintained under repeated differentiation by vector

fields tangent to Z; in particular they are smooth away from Z. Lagrangian distributions, and their generalizations, play a central role in modern PDE theory, see e.g. [5].

If X_0 is not compact, one can study decay/growth properties of distributions at 'infinity' in addition to studying their smoothness properties. Thus, the lack of rapid decay at infinity can be considered a 'singularity' and studied via microlocal analysis. Since one needs some structure at infinity, even to make sense of 'rapid decay', it is more natural to work on compact manifolds with boundary (or corners) which arise by the compactification of such X_0 and study singularities at the boundary. On manifolds with boundary, X, one can either introduce Legendre distributions from the symplectic (or really contact) point of view, as traditionally done for Lagrangian distributions, or instead simply write down such distributions and 'work backwards'. Since the former point of view, which is certainly 'neater', has been discussed in detail in [9], we follow the second approach. This should quickly make it clear that many familiar functions fall in the class of Legendre distributions.

Thus, let X be a compact manifold with boundary, and x a boundary defining function. Legendre distributions on X are functions which are smooth in the interior of X and have specific types of singularities at the boundary, which we shall now describe. The simplest example is a function of the form

$$(1.1) \qquad u_1 = x^q e^{i\phi(x,y)/x} a(x,y), \text{ with } \phi, a \text{ smooth on } X.$$

Here (x,y) are coordinates on X, where $y = (y_1, \ldots, y_{n-1})$ restrict to coordinates on ∂X. There is no loss of generality in assuming that ϕ depends only on y. Another example is a function

$$x^q V(x, y/x),$$

where $V(x,w)$ is smooth in x and Schwartz in w. More generally, if $y = (y', y'')$ is a splitting of the coordinates then a function of the form

$$(1.2) \qquad u_2 = x^q V(x, y'/x, y'')$$

with $V(x, w', y'')$ smooth and Schwartz in w', is a Legendre distribution. Distributions such as u_1 arise, for example, as plane waves in scattering theory, while distributions like u_2 arise as potentials in many-body scattering, as we explain below. This already makes it clear that we will need a combination of these types of distributions, e.g. to understand perturbed plane waves, in a precise manner, in many-body scattering.

These two types of Legendre distributions can be given a uniform description in terms of Legendre submanifolds. Let U be an open subset of ∂X, and consider the space $U \times \mathbb{R} \times \mathbb{R}^{n-1}$ with coordinates y, τ, μ. The form $\chi = d\tau + \mu \cdot dy$ is a contact form on $U \times \mathbb{R} \times \mathbb{R}^{n-1}$; that is, $\chi \wedge (d\chi)^{n-1}$ never vanishes. (Here we work in local coordinates, but this has an invariant geometric description in terms of the scattering cotangent bundle; see [9, 3] for a detailed description, and the next section for a brief summary.) Recall that a Legendre distribution is a submanifold of maximal dimension (equal to $n-1$) on which the contact form vanishes. The Legendre submanifold associated with u_1 is

$$G_1 = \{(y, \tau = -\phi(y), \mu = d_y\phi(y))\}.$$

This submanifold determines ϕ. For each $m \in \mathbb{R}$, the class $I_{\text{sc}}^m(X, G_1)$ is the class of functions u_1 of the form (1.1) with $q = m + n/4$ and $a \in \mathcal{C}^\infty(X)$ arbitrary.

The second example, u_2, can be written in terms of the Fourier transform in the w variable as

$$u_2 = x^q \int e^{iy' \cdot \eta'/x} \hat{V}(x, \eta', y'') d\eta'. \tag{1.3}$$

The Legendre submanifold associated with u_2 is

$$G_2 = \{(y, \tau, \mu) \mid y' = 0, \tau = 0, \mu'' = 0\}.$$

If $\dim y' = k$ then the class $I_{\text{sc}}^m(X, G_2)$ is the class of functions of the form (1.3) with $q = m + n/4 - k/2$ and with $\hat{V}$ Schwartz in the second variable.

More generally, any Legendre submanifold L of $U \times \mathbb{R} \times \mathbb{R}^{n-1}$ has a local parametrization, that is, a function $\phi(y, v)$, where $y \in U$ and $v \in \mathbb{R}^p$ such that locally

$$L = \{(y, \tau, \mu) \mid \exists\, (y, v) \text{ such that } \tau = -\phi(y, v), \mu = d_y \phi(y, v) \text{ and } d_v \phi(y, v) = 0\},$$

and

$$\text{for } 1 \le i \le p, \quad d\left(\frac{\partial \phi}{\partial v_i}\right) \text{ are linearly independent.} \tag{1.4}$$

Condition (1.4) ensures that the map

$$\{(y, v) \mid d_v \phi(y, v) = 0\} \to \{(y, \tau, \mu) \mid \tau = -\phi, \mu = d_y \phi\} \tag{1.5}$$

is a diffeomorphism. The class $I_{\text{sc}}^m(X, L)$ of Legendre distributions are defined as a finite sum of terms of the form

$$x^{m+n/4-p/2} \int_{\mathbb{R}^p} e^{i\phi(y,v)/x} a(x, y, v) dv,$$

where ϕ locally parametrizes L and a is smooth, with compact support in v. (We also allow the case that ϕ is a linear function of v and a is Schwartz in v; these are called 'extended Legendrian distributions' in [2].) It is not hard to see that the above examples are instances of this framework.

The *microsupport* of u is the closed subset of L corresponding, under (1.5), to the set

$$\{(y, v) \mid d_v \phi = 0 \text{ and there is no neighbourhood of } (0, y, v) \text{ in which } a \text{ is } O(x^\infty)\}.$$

By a partition of unity we can always write u as a finite sum of terms each having microsupport as small as desired.

Such distributions turn up naturally in scattering theory. For example, let X be the radial compactifiction of $\mathbb{R}^n$, and let z be a linear coordinate on $\mathbb{R}^n$. Consider the function $e^{-iz \cdot k}$, $k \in \mathbb{R}^n$ which is a generalized eigenfunction of Δ with eigenvalue $|k|^2$. In inverted polar coordinates $(x, \hat{z})$, $x = |z|^{-1}$, $z = \hat{z}/x$, this function takes the form $e^{i\hat{z} \cdot k/x}$, which is a Legendre distribution of the first type. An example of a Legendre distribution of the second type is a function of some subset of the z variables: let $z = (z', z'')$ and let V be a Schwartz function of z'. Then

$$V(z') = (2\pi)^{-k} \int e^{i\hat{z}' \cdot \zeta'/x} \hat{V}(\zeta') d\zeta'$$

is a Legendre distribution of the second type. Such functions appear naturally in the quantum N-body problem for example.

However, very frequently one comes across functions which are more complicated, and associated to more than one Legendre submanifold. For example, consider the kernel of the outgoing resolvent $R(\sigma + i0)$ of the Laplacian on $\mathbb{R}^3$. This is

$$G(\sigma, z, z') = \frac{1}{4\pi} \frac{e^{i\lambda|z-z'|}}{|z-z'|}, \quad \lambda = \sqrt{\sigma} > 0.$$

Let us multiply by a function $\chi(|z-z'|)$ which is smooth, vanishes near $|z-z'| = 0$ and is $\equiv 1$ for $|z-z'| > c$ to get rid of the interior singularity. (The difference is the kernel of a pseudodifferential operator, which is well understood.) Consider the resulting function, for fixed $\lambda > 0$, as a function on the radial compactification $\tilde{X}$ of $\mathbb{R}^6$. Let $C \subset \partial\tilde{X}$ be the boundary of the diagonal $z = z'$. Away from C, $|z-z'|$ is a smooth function homogeneous of degree zero, so can be written $\phi(y)/x$, where y is a coordinate on $\partial\tilde{X}$ and $x = (|z|^2 + |z'|^2)^{-1/2}$ is the reciprocal of the Euclidean distance from the origin. Thus the kernel is a Legendre distribution associated to a Legendre submanifold L_1 of the first type in this region. On the other hand, if we restrict to a region $|z-z'| < R$ near the diagonal then it looks like a Legendre distribution associated with a Legendre submanifold L_2 of the second type: a smooth function of the three variables $z-z'$. Clearly it is simultaneously associated to both of these Legendrian submanifolds. The geometry of these submanifolds is such that L_1 is a manifold with boundary, which intersects L_2 cleanly at $L_1 \cap L_2 = \partial L_1$.

There are (at least) two ways of looking at such distributions, and consequently, two classes of distributions associated to (L_2, L_1). The first way is to define *intersecting Legendre distributions* as was done in [2] (which is a routine generalization of the class of intersecting Lagrangian distributions to the Legendre setting). One defines a local parametrization of (L_2, L_1) near a point $q \in L_1 \cap L_2$ to be a function $\phi(y, v, s)$, where $v \in \mathbb{R}^p$ and $s \in [0, \infty)$, such that ϕ parametrizes L_1 in the sense analogous to (1.5) with both v and s taken as parameters, while $\phi(y, v, 0)$ parametrizes L_2. The nondegeneracy condition (1.4) is replaced by

$$(1.6) \qquad d\left(\frac{\partial\phi}{\partial s}\right), d\left(\frac{\partial\phi}{\partial v_i}\right) \text{ and } ds \text{ are linearly independent.}$$

Then an expression of the form

$$(1.7) \qquad x^{m+n/4-(p+1)/2} \int \int_0^\infty e^{i\phi(y,v,s)/x} a(x, y, v, s) \, ds \, dv,$$

is a Legendre distribution of order m associated to L_1 (e.g. when a is supported in $s \geq \epsilon > 0$) and of order $m + 1/2$ at $L_2 \setminus L_1$ (to see this, multiply and divide by $d_s\phi/x$ and then integrate by parts in s to get a boundary term at $s = 0$). The class $I_{\mathrm{sc}}^m(X, (L_2, L_1))$ is defined to be those functions $u = u_1 + u_2 + u'$, where $u_1 \in I_{\mathrm{sc}}^m(X, L_1)$, $u_2 \in I_{\mathrm{sc}}^{m+1/2}(X, L_2)$ and u' given by a finite sum of terms of the form (1.7).

The second way, when L_2 arises from an embedded submanifold $C \subset \partial X$ as above, involves blowing up the submanifold C and defining *fibred Legendrian distributions*. Let

$$Y = [X; C]$$

be the manifold with codimension 2 corners obtained by real blowup of C. Y has two boundary hypersurfaces: one which is the lift of ∂X, which we call the 'main face', and a new hypersurface arising from the blowup of C which we call the 'front
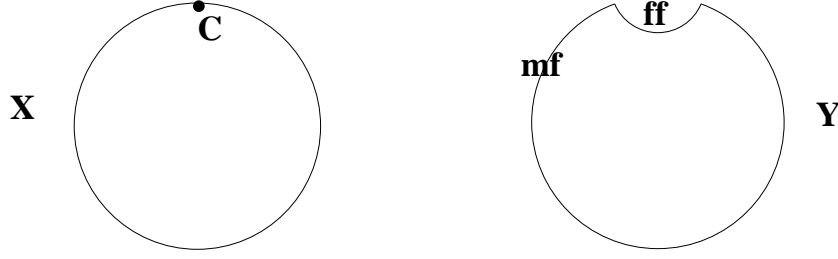
FIGURE 1. Blowing up $C \subset \partial X$ to produce Y.

face'. These will be abbreviated mf and ff, respectively. Then $I^{m+1/2}_{\text{sc}}(X, L_2)$ is identical with functions of the form $x^{m+1/2-k/2+n/4}c$, where c is smooth on Y and vanishes to infinite order at mf, and k is the dimension of the fibers of the blow-down map, i.e. it is the codimension of C in ∂X. Suppose L_1 is a Legendrian with boundary meeting L_2 cleanly at $\partial L_1 = L_1 \cap L_2$. Let (y', y'') be coordinates on ∂X which define C as $\{x = 0, y' = 0\}$. If we assume that $L_1 \cap L_2$ has a full rank projection to C, then this implies that near $q \in \partial L_1$, there must be one of the y' coordinates, say y'_k, with a nonzero differential at q (this is shown in the following section). Then, there is a parametrization of L_1 near q of the form

$$\phi(y, v)/x = y_k \tilde{\phi}(y'/y_k, y_k, y'', v)/x, \quad v \in \mathbb{R}^p.$$

We define the class of fibred Legendrians of order (m, r) associated to L to be those functions $u = u_1 + u_2 + u'$, where $u_1 \in I^m_{\text{sc}}(X, L_1)$ and supported away from C, $u_2 \in I^r_{\text{sc}}(X, L_2)$ and u' given by a finite sum of terms of the form

$$(1.8) \qquad y_k^{r+\frac{n}{4}-\frac{k}{2}} \left(\frac{x}{y_k}\right)^{m+\frac{n}{4}-\frac{p}{2}} \int e^{iy_k \tilde{\phi}(\frac{y'}{y_k}, y_k, y'', v)/x} a(y_k, \frac{y'}{y_k}, \frac{x}{y_k}, y'', v)dv.$$

Again a is assumed to be smooth and compactly supported in all variables. Note that this now means that a is smooth in terms of the differentiable structure on Y, rather than on X.

The purpose of this paper is to clarify the relation between these two spaces $I^m_{\text{sc}}(X, (L_2, L_1))$ and $I^{m,m+1/2}_{\text{s}\Phi}(Y, L)$ when L_2 is the Legendre submanifold associated to C as above.

2. Invariant Description

To describe the situation more invariantly, let X be a manifold with boundary, x a boundary defining function, and let C be a closed embedded submanifold of ∂X. We denote the interior of X by X°. Let $p \in C$ and let $(x, y) = (x, y_1, \ldots, y_{n-1})$ be coordinates near p, where x is a defining function for ∂X and $C = \{x = 0, y' = 0\}$ near p; here $y' = (y_1, \ldots, y_k)$. Then X is naturally equipped with its scattering cotangent bundle, $^{\text{sc}}T^*X$. One way to describe $^{\text{sc}}T^*X$ is that its smooth sections are spanned, over $\mathcal{C}^\infty(X)$, by one-forms of the form $d(\phi/x)$, $\phi \in \mathcal{C}^\infty(X)$. In particular, $^{\text{sc}}T^*_{X^\circ}X$ is naturally identified with T^*X°. A local basis for $^{\text{sc}}T^*X$ near p is given by $dx/x^2 = -d(1/x)$ and dy_i/x. Thus, a point $q \in {}^{\text{sc}}T^*X$ may be written

$$(2.1) \qquad q = \tau \frac{dx}{x^2} + \sum_i \mu_i \frac{dy_i}{x},$$

and this gives local coordinates (x, y, τ, μ) on $^{sc}T^*X$, where (τ, μ) are linear co-ordinates on each fibre. Moreover, $^{sc}T^*_{\partial X}X$ is naturally equipped with a contact structure via a contact form χ induced by the symplectic structure of T^*X°, in a similar way to that in which a contact form is induced on the cosphere bundle $S^*X = (T^*X \setminus 0)/\mathbb{R}^+$. In the local coordinates given by (2.1), this form is equal to $\chi = d\tau + \mu \cdot dy$, as in the previous section.

There is a well-defined 'scattering conormal bundle' over C, denoted $^{sc}N^*(C; X)$, which is defined as the span of $d(\phi/x)$ (inside $^{sc}T^*_C X$) for all $\phi \in \mathcal{C}^\infty(X)$ which vanish on C. In the local coordinates (x, y', y''), ϕ can be written as $\phi = \sum_j y'_j a_j + x a_0$, so $^{sc}N^*(C; X)$ is spanned by the dy'_j/x, i.e. the dimension of each fibre is is the codimension of C in ∂X. It is easy to see that $^{sc}N^*(C; X)$ is in fact a Legendre submanifold of $^{sc}T^*_{\partial X}X$.

Let $Y = [X; C]$ be the blow-up of X at C, and let $\beta : Y \to X$ be the blow-down map. Let mf denote the lift of ∂X to Y, and let ff denote the front face of the blow-up, i.e. the lift of C. Then ff has a natural fibration over C given by the blow-down map: $\beta|_{ff} : ff \to C$. As discussed in [3], this defines a structure algebra $\mathcal{V}_{s\Phi}(Y)$ of vector fields, and more importantly for us, a corresponding replacement of the standard cotangent bundle, namely the scattering fibred cotangent bundle $^{s\Phi}T^*Y$. Sections of $^{s\Phi}T^*Y$ are spanned, over $\mathcal{C}^\infty(Y)$, by $d(\phi/x)$ where $\phi \in \mathcal{C}^\infty(Y)$ is constant on the fibers of the fibration. Such a setting is a natural generalization (to manifolds with corners) of the fibred cusp Lie algebra introduced by Mazzeo and Melrose on manifolds with a fibred boundary [7]. Then $^{s\Phi}T^*_{mf}Y$ has a natural contact form which degenerates at the corner $^{s\Phi}T^*_{mf \cap ff}Y$. In this setting, $^{s\Phi}T^*Y$ is just the pull-back of $^{sc}T^*X$ by the blow-down map β, and we denote the induced map by $\tilde{\beta} : {}^{s\Phi}T^*Y \to {}^{sc}T^*X$. Thus, local coordinates on $^{s\Phi}T^*Y$ near the boundary consist of local coordinates on Y together with the functions τ and μ lifted from $^{sc}T^*X$. Moreover, the contact form on $^{s\Phi}T^*_{mf}Y$ is just the pull-back of $d\tau + \mu \cdot dy$ by $\tilde{\beta}$. Given a Legendre submanifold $L \subset {}^{s\Phi}T^*_{mf}Y$ which is transversal to the corner mf $\cap$ ff and satisfies a compatibility condition with the fibration, the class of Legendre distributions associated to L was defined in [3]; in the case of interest here, the definition (1.8) above suffices. In particular, if L has full rank projection to mf, the compatibility condition is automatically satisfied, and such a Legendre distribution is simply of the form $e^{i\phi/x}a$ where $\phi \in \mathcal{C}^\infty(Y)$ is constant along the fibers of $\beta|_{ff}$, and $a \in \mathcal{C}^\infty(Y)$.

3. Main Results

To describe our main results, we let L_2 be the scattering conormal bundle $^{sc}N^*(C; X)$ as above, and suppose that $L_1 \subset {}^{sc}T^*X$ is a Legendre submanifold with boundary which intersects L_2 cleanly in $\partial L_1 = L_2 \cap L_1$. In addition, we assume that $L_2 \cap L_1$ has a full rank projection to C. Thus, in local coordinates in $^{sc}T^*_{\partial X}X$, $L_2 = \{y' = 0, \ \tau = 0, \ \mu'' = 0\}$. We shall prove in Lemma 5.1 that these assumptions imply that L_1 also intersects $^{sc}T^*_C X$ cleanly, with intersection $L_2 \cap L_1$, and in particular that at any $q \in L_1$, the pull-back of dy'_j to L_1 does not vanish for some j. Without loss of generality we may assume that the restriction of dy'_k to L_1 does not vanish at q; thus, near q, we may assume that $y'_k \geq 0$ is a boundary defining function for L_1. This implies that under the blow-down map $\beta^* : {}^{s\Phi}T^*_{mf}Y \to {}^{sc}T^*_{\partial X}X$, L_1 lifts to a Legendre submanifold L which is transversal to $^{s\Phi}T^*_{mf \cap ff}Y$ and has a full rank projection to C.

Let $\tilde{L}_1$ be an extension of L_1 across its boundary to an (open) Legendre submanifold. Our first result is a characterization of $I_{\mathrm{sc}}^m(X, (L_2, L_1))$ in terms of $I^m(X, \tilde{L}_1)$.

Theorem 3.1. *Let $\alpha \in \mathcal{C}^\infty(\mathbb{R})$, with $\alpha(t)$ identically 1 for $t > 1$, identically 0 for $t < 0$. Let U be a neighborhood of $q \in L_2 \cap L_1$ in $^{sc}T_{\partial X}^* X$ such that y_k' is a defining function for ∂L_1 in $L_1 \cap U$. Intersecting Legendre distributions $u \in I_{sc}^m(X, (L_2, L_1))$ microsupported in U may be represented as*

$$(3.1) \qquad \alpha\Big(\frac{y_k}{x}\Big) u_1 + u_0,$$

where $u_1 \in I_{sc}^m(X, \tilde{L}_1)$ and $u_0 \in I_{sc}^{m+1/2}(X, L_2)$. Conversely, any such function is in $I_{sc}^m(X, (L_2, L_1))$.

Given this theorem we can rather easily understand the relation between the spaces $I_{\mathrm{sc}}^m(X, (L_2, L_1))$ and $I_{\mathrm{s}\Phi}^{m,m+1/2}(X, L)$.

Theorem 3.2. *The space $I_{sc}^m(X, (L_2, L_1))$ is a proper subset of $I_{s\Phi}^{m,m+1/2}(X, L)$. In particular, $I_{s\Phi}^{m,m+1/2}(X, L)$ is invariant under multiplication by $\mathcal{C}^\infty(Y)$ functions, while $I_{sc}^m(X, (L_2, L_1))$ is not.*

Remark. One can show that the algebraic $\mathcal{C}^\infty(Y)$ module generated by the space $I_{\mathrm{sc}}^m(X, (L_2, L_1))$ is dense in $I_{\mathrm{s}\Phi}^{m,m+1/2}(X, L)$. Hence, one could say that $I_{\mathrm{s}\Phi}^{m,m+1/2}(X, L)$ is generated over $\mathcal{C}^\infty(Y)$ by $I_{\mathrm{sc}}^m(X, (L_2, L_1))$.

Remark. Guillemin and Uhlmann [1] and Joshi [6] have defined paired Lagrangian distributions of independent orders (m, r) associated to a pair (L_2, L_1) with the geometry as described above. Using the Fourier transform and a local identification of neighbourhoods of $p \in \partial X$ with cones in $\mathbb{R}^n$, one can define paired Legendre distributions. In our setting these are defined by

$$(3.2) \qquad x^{m+n/4-(p+1)/2} \int \int_{-\infty}^{\infty} s_+^{r-(m+1/2)} e^{i\phi(y,v,s)/x} a(x, y, v, s) \, ds \, dv,$$

i.e. the distribution $s_+^{r-(m+1/2)}$ replaces the Heaviside step function $H(s) = s_+^0$. It is not hard to show that Theorem 3.2 holds in this setting with distributions of the form (3.2) forming a proper subset of the class $I_{\mathrm{s}\Phi}^{m,r}(X, L)$.

Remark. It is very natural to assume that $L_1 \cap L_2$ is codimension one in L_2 and in L_1 since this appears naturally in real principal type propagation. However, one can also consider the case of higher codimension intersections. In the case of Lagrangian manifolds, such intersections were studied by Guillemin and Uhlmann [1]. They essentially define the associated class of distributions by blowing up the intersection $L_2 \cap L_1$. Note that if the intersection has codimension one, the blow-up divides L_1 into two manifolds with boundary, each with boundary $L_2 \cap L_1$, hence resulting exactly in the setting discussed above. In our case thus the most natural definition of such intersecting Legendre distributions associated to an intersecting pair (L_2, L_1) with $L_2 \cap L_1$ having codimension greater than one is via the fibred scattering structure rather than directly by oscillatory integrals. In particular, if $L_2 = {}^{sc}N^*(C; X)$, L_1 is the zero section of $^{sc}T_{\partial X}^* X$, the codimension of the intersection $L_2 \cap L_1$ (in L_2 and in L_1) is given by the dimension of the fibers of $^{sc}N^*(C; X) \to C$, namely by the codimension of C in ∂X. The natural definition of the class $I_{\mathrm{sc}}^m(X, (L_2, L_1))$ of distributions associated to the intersecting Legendre pair (L_2, L_1) is functions of the form $x^{m+\dim X/4}\alpha(|y'|/x)f + g$, $f \in \mathcal{C}^\infty(X)$, $g \in \mathcal{C}^\infty([X; C])$ with infinite order vanishing on mf.

4. An example

Although the general case is hardly more complicated, for the sake of clarity we first consider the case when $\dim X = 2$, and $C \subset \partial X$ is a point $x = 0$, $y = 0$, and L_1 is the zero section of ${}^{\mathrm{sc}}T^*X$ in $y \geq 0$. The lift L of L_1 to ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}Y$ is the zero section then. Thus, fibred Legendrians in $I^{m,m+1/2}_{\mathrm{s}\Phi}(X, L)$ are $\mathcal{C}^\infty$ functions on $Y = [X; C]$, multiplied by powers of the boundary defining function, i.e. functions of the form $x^{m+1/2}c$, $c \in \mathcal{C}^\infty(Y)$. Let us show that elements of $I^m_{\mathrm{sc}}(X, (L_2, L_1))$ are functions of the form $x^{m+1/2}c$, where $c \in \mathcal{C}^\infty(Y)$ is of the form

$$(4.1) \qquad c = \alpha\Big(\frac{y}{x}\Big)f(x, y) + g\Big(x, \frac{y}{x}\Big).$$

Here f is smooth, α is as in Theorem 3.1, and $g(x, Z)$ is a smooth function of x taking values in Schwartz functions of Z. Thus, c is locally an arbitrary element of $\mathcal{C}^\infty(Y)$ away from the corner, while at the corner, its Taylor series is restricted so that it only has terms of the form $y^j(x/y)^k$ with $j \geq k$.

An intersecting Legendre distribution, $u \in I^m_{\mathrm{sc}}(X, (L_2, L_1))$, associated to these Legendrians is one which can be written as $u = u_0 + u_1 + u'$, with $u_0 \in I^{m+1/2}_{\mathrm{sc}}(X, L_2)$, $u_1 \in I^m_{\mathrm{sc}}(X, L_1)$, and u' of the form

$$(4.2) \qquad x^{m-1/2} \int_0^\infty \int e^{i\zeta \cdot (y - \bar{y})/x} a(x, y, \zeta, \bar{y}) \, d\zeta \, d\bar{y},$$

where a has compact support in $x, y, \bar{y}$, and is Schwartz in ζ. Directly from their definition, Legendre distributions in $I^m_{\mathrm{sc}}(X, L_1)$ are in fact of the form $x^{m+1/2}f(x, y)$, where f is rapidly decreasing at the boundary wherever $y < 0$. On the other hand, it is easy to see that Legendrians in $I^{m+1/2}_{\mathrm{sc}}(X, L_2)$ are in $x^{m+1/2}\mathcal{C}^\infty(Y)$, vanishing to infinite order off the front face. Indeed, by definition, such a distribution u_0 can be written as

$$x^{m+1/2} \int e^{i\zeta y/x} a(x, y, \zeta) \, d\zeta,$$

with a compactly supported in x, y, rapidly decreasing in ζ. But this is just Fourier transform in ζ, hence the result is of the form $x^{m+1/2}b(x, y, y/x)$, with compact support in x, y, rapid decay in y/x, which means that $u_0 \in x^{m+1/2}\mathcal{C}^\infty(Y)$ vanishing to infinite order off the front face. Conversely, if $u_0 = x^{m+1/2}v$, $v \in \mathcal{C}^\infty(Y)$ with infinite order vanishing on mf, then modulo a Schwartz function, v is a $\mathcal{C}^\infty$ function of x, y and y/x which is compactly supported in x and y, and Schwartz in y/x. Thus, we can write u_0 (modulo $\dot{\mathcal{C}}^\infty(X)$) as a Legendre function associated to L_2 by writing v as the inverse Fourier transform of its Fourier transform in y/x:

$$u_0(x, y) = (2\pi)^{-1} x^{m+1/2} \int e^{i\zeta y/x} \hat{v}(x, \zeta) \, d\zeta,$$

modulo $\dot{\mathcal{C}}^\infty(X)$, which is the definition of a Legendre distribution associated to L_2.

Hence the real question is whether pieces as in (4.2) are of the form (4.1), and conversely, whether such functions supported near the corner can be written as in (4.2) modulo $I^{m+1/2}_{\mathrm{sc}}(X, L_2) + I^m_{\mathrm{sc}}(X, L_1)$.

Start with the former, i.e. consider an oscillatory integral of the form (4.2). The phase function $\phi = \zeta \cdot (y - \bar{y})$, is stationary with respect to ζ if $y = \bar{y}$, and it is stationary with respect to $\bar{y}$ if $\zeta = 0$. Thus, L_1' corresponds to the set $\zeta = 0$, $y = \bar{y}$, $\bar{y} \geq 0$, in the parameter space, while L_2 corresponds to $\bar{y} = 0$, $y = 0$. We may,

modulo a Schwartz error, assume that a is independent of $\bar{y}$. The reason is that we can expand a as a Taylor series around $y = \bar{y}$:

$$a(x, y, \bar{y}, \zeta) = a(x, y, y, \zeta) + (y - \bar{y})a_1(x, y, \bar{y}, \zeta).$$

Since $(y - \bar{y}) = \partial_\zeta \phi$, so by integrating by parts to get rid of this factor on the second term we gain a power of x. Thus, iterating the procedure and using asymptotic completeness we only have to deal with the first term.

We start with the ζ integral, which simply takes a Fourier transform of a in ζ and evaluates the result at $-(y - \bar{y})/x$. We write $\hat{a}$ for the Fourier transform; so $\hat{a}$ is a Schwartz function in its third variable (since it is the Fourier transform of a Schwartz function in that variable). Indeed, we can arrange that a is such that its Fourier transform in ζ has compact support (and is smooth). Letting $Z = y/x$ and $Z' = (y - \bar{y})/x$ and changing variables in the $\bar{y}$ integral we obtain

$$(4.3) \quad \begin{aligned} x^{m-1/2} \int_0^\infty \hat{a}(x, y, -(y - \bar{y})/x)d\bar{y} &= x^{m+1/2} \int_{-\infty}^Z \hat{a}(x, y, -Z')dZ' \\ &= x^{m+1/2} \int_{-\infty}^Z \left[d(x, y)\alpha'(Z') + b(x, y, Z') \right] dZ' \end{aligned}$$

where $d(x, y) = \int_{\mathbb{R}} \hat{a}(x, y, -Z')dZ'$ is chosen so that the integral of b in Z' (over $\mathbb{R}$) vanishes. Thus, the integral of b from $-\infty$ to Z is a Schwartz function of Z, so we have written this in the form

$$d(x, y)\alpha(Z) + g(x, xZ, Z).$$

Expanding g as a Taylor series in the second variable we see that we have expressed u in the form (4.1).

Conversely, let u be of the form (4.1). The g term is associated to L_2, so we only need deal with the first term. In fact, it is clear that the class $I_{\mathrm{sc}}^m(X, (L_2, L_1))$ is invariant under multiplication by smooth functions, so we need only treat the α function. To do this, we write

$$(4.4) \quad \begin{aligned} \alpha(Z) &= \int_{-\infty}^Z \alpha'(Z')dZ' \\ &= x^{-1} \int_0^\infty \alpha'\left(\frac{y - \bar{y}}{x}\right)d\bar{y} \\ &= x^{-1} \int d\zeta \int_0^\infty e^{i(y-\bar{y})\zeta/x}\widehat{\alpha'}(\zeta)d\bar{y}d\zeta \end{aligned}$$

which is of the right form since α', and therefore also its Fourier transform, are Schwartz.

5. The general case

Let X, C, Y, L_2, L_1, $\tilde{L}_1$, and L be as in section 3, and let $q \in \partial L_1$. Let $\tilde{L}_1$ be a Legendrian extension of L_1 to a submanifold without boundary across L_2. In local coordinates, $C = \{x = 0, y' = 0\}$ and $L_2 = \{x = 0, y' = 0, \mu'' = 0\}$. First we prove a statement asserted just before the main results from section 3.

Lemma 5.1. *There is a y' coordinate, which may be taken to be y_k without loss of generality, whose differential restricted to L_1 does not vanish at q.*

Proof. Let $q \in L_2 \cap L_1$, and let $W = T_q(L_2 \cap L_1) \subset Z = T_q{}^{\mathrm{sc}}T^*_{\partial X}X$. Moreover, let $V_j = T_q L_j \subset Z$. The fact that a subspace V of Z is Legendre means that both χ and $d\chi$ vanish on it identically, i.e. $\chi(v) = 0$, $d\chi(v, v') = 0$ for all $v, v' \in V$, and V is maximal, i.e. $(\dim Z - 1)/2 = n - 1$ dimensional, $n = \dim X$, with this property. Note that $d\chi$ is non-degenerate on $\operatorname{Ker} \chi \subset Z$, i.e. it is a symplectic form on this vector space.

Now, both χ and $d\chi$ vanish on W since $W \subset V_2$. Let W' denote the subspace of $\operatorname{Ker} \chi$ which annihilates W, i.e. $W' = \{w' \in Z : \chi(w') = 0, \, d\chi(w', w) = 0 \text{ for all } w \in W\}$. Then any Legendre subspace $V \supset W$ of Z satisfies $V \subset W'$ since $V \subset \operatorname{Ker} \chi$ and $d\chi(v, w) = 0$ for all $w \in W \subset V$. Note that W has codimension 2 in W' (since W has codimension 1 in the Legendre subspace V_2). Thus, W'/W is a 2-dimensional vector space, and $d\chi$ descends to a symplectic form on it. The image V' of a Legendre subspace $V \supset W$ in W'/W is Lagrangian with respect to this form. There is a one-dimensional family of such Lagrangian subspaces; the image of V_2 is one of them. Indeed, given any non-zero element u of W'/W, there is a unique Lagrangian subspace of W'/W which includes u, namely the span of u. This then determines a unique Legendre subspace V of W' with $W \subset V$.

We claim that W' is not a subspace of $T_q{}^{\mathrm{sc}}T^*_C X$. Indeed, suppose otherwise, i.e. that $W' \subset T_q{}^{\mathrm{sc}}T^*_C X$. The hypothesis on the full rank projection of $L_2 \cap L_1$ means that dy''_j are independent on W. The corresponding Hamilton vectors $\partial_{\mu''_j}$ under $d\chi$ in $\operatorname{Ker} \chi$ are tangent to ${}^{\mathrm{sc}}T^*_C X$, hence in $T_q{}^{\mathrm{sc}}T^*_C X \cap \operatorname{Ker} \chi$. Thus, they span a $(\dim C)$-dimensional subspace T of this space. If f is a nonzero linear combination of the functions y''_j, then df does not vanish on W, which implies that $d\chi(H_f, \cdot)$ does not vanish on W. Hence $H_f \notin W'$, which means that T and W' have trivial intersection. But this is a contradiction: by dimension counting, the codimension of W' in $T_q{}^{\mathrm{sc}}T^*_C X \cap \operatorname{Ker} \chi$ is $\dim C - 1$. Hence W' is not a subspace of $T_q{}^{\mathrm{sc}}T^*_C X$.

Thus dy'_j cannot all vanish identically on W'. Since they all vanish identically on V_2, this is the only Legendre subspace with this property. By the clean intersection assumption, $V_2 \cap V_1 = W$, i.e. V_2 and V_1 are not the same. Hence the dy'_j do not all vanish on V_1, i.e. the pull-back of dy'_j to L_1 at q is non-zero for some j. By relabelling the coordinates, we may assume that dy'_k is non-zero. $\qquad\square$

Since ∂L_1 has a full rank projection to C, the span of the pull-back of the differentials dy'_j to L_1 at q is exactly one-dimensional. By a linear change of the y coordinates we may assume that $dy'_1, \dots, dy'_{k-1}$ pull back to 0 at q. Then standard contact arguments show that y'_k, y'' and $v = (\mu'_1, \dots, \mu'_{k-1})$ give local coordinates on $\tilde{L}_1$ near q, and $L_2 \cap L_1$ is defined by $y_k = 0$ in these coordinates. By switching the sign of y_k if necessary, we may also assume that microlocally L_1 lies in $y_k \geq 0$. Expressing $\tilde{y} = (y_1, \dots, y_{k-1})$ as $\tilde{y} = \tilde{Y}(y_k, y'', v)$, τ as $\tau = T'(y_k, y'', v)$ on $\tilde{L}_1$, it follows that a local nondegenerate parameterization of $\tilde{L}_1$ is given by

$$(5.1) \qquad \phi(y, v) = -T' + v \cdot (\tilde{y} - \tilde{Y}), \quad v \in \mathbb{R}^{k-1},$$

while a local parametrization of L is given by the same function for $y_k/x > C$. A local nondegenerate parametrization of (L_2, L_1) is given by

$$(5.2) \qquad \psi(y, v, \zeta, \bar{y}) = -T' + v \cdot (\tilde{y} - \tilde{Y}) + \zeta(y_k - \bar{y}), \quad \bar{y} \geq 0.$$

Note that $\tilde{y}$ when $y_k = 0$, so in fact $\tilde{Y} = y_k Y$, and similarly $T' = y_k T$ (since $\tau = 0$ on L_2).

Proof of Theorem 3.1.   Write $u \in I_{sc}^m(X, (L_2, L_1))$ in terms of the phase function ψ from (5.2). We may then run the argument in the example of the previous section in the variables $(x, y_k, \bar{y}, \zeta)$ to show that u can be written in the form (3.1). To prove the converse, write u_1 with respect to the phase (5.1) and express α as in (4.3) to obtain an expression involving the phase function ψ from (5.2), which is manifestly an element of $I_{sc}^m(X, (L_2, L_1))$.

Proof of Theorem 3.2.   First we show that $I_{sc}^m(X, (L_2, L_1))$ is contained in $I_{s\Phi}^{m,m+1/2}(X, L)$. Theorem 3.1 to write $u \in I_{sc}^m(X, (L_2, L_1))$ in the form (3.1). In terms of this representation, u_0 is $x^{m+n/4-(k-1)/2}$ times an element of $\mathcal{C}^\infty(Y)$ which vanishes to all orders at the main face, so this is certainly an element of $I_{s\Phi}^{m,m+1/2}(X, L)$ (in fact, $I_{s\Phi}^{r,m+1/2}(X, L)$ for any r). On the other hand, u_1 is in $I_{sc}^m(X, \tilde{L}_1)$ so can be written with respect to the phase function ϕ from (5.1). But this is also a phase function for L, and multiplication by a suitable α means it is now supported in $y_k'/x > C$, so $\alpha \cdot u_1$ is also in $I_{s\Phi}^{m,m+1/2}(X, L)$.

To show that the inclusion is proper, we write u in terms of a reduced symbol and show that its Taylor series is restricted at $x/y_k' = y_k' = 0$, that is, at the intersection of the front face and the main face on Y. Thus, we can write u in the form (1.8) where a only depends on $y_k, x/y_k', y''$ and v; also note that when $r = m + 1/2$ and $p = k - 1$, then the power of y_k' outside the integral vanishes. By the symbol calculus of [4], the reduced symbol is then determined to all orders in Taylor series at mf $\cap$ ff by u. Using the description given by Theorem 3.1, we see that the u_0 term has trivial Taylor series at $x/y_k' = y_k' = 0$. The $\alpha \cdot u_1$ term has the property that the symbol for u_1 is smooth in the variables x and y_k', so the Taylor series of a as a function of x/y_k' and y_k' has the property

(5.3)   The coefficient of $(x/y_k')^j y_k^l$ vanishes whenever $l < j$.

Thus, the sum of the two terms u_0 and $\alpha \cdot u_1$ has property (5.3). It is clear that this property is not invariant under multiplication by smooth functions of y_k' and x/y_k'. However, the space $I_{s\Phi}^{m,m+1/2}(X, L)$ is by its definition invariant under $\mathcal{C}^\infty(Y)$. So $I_{sc}^m(X, (L_2, L_1))$ is a strictly smaller space than $I_{s\Phi}^{m,m+1/2}(X, L)$. This completes the proof of the Theorem.

Remark. The absence of terms (5.3) in the symbol of $I_{sc}^m(X, (L_2, L_1))$ is reflected in the symbol calculus for $u \in I_{sc}^m(X, (L_2, L_1))$. That is, if the symbol $\sigma_{L_1}^m(u)$ of u on L_1 vanishes, then $u \in I_{sc}^{m-1}(X, (L_2, L_1)) + I_{sc}^{m+1/2}(X, L_2)$ (see [8], equation (5.2)). On the other hand, if the symbol of $u \in I_{s\Phi}^{m,m+1/2}(X, L)$ at L vanishes, then $u \in I_{s\Phi}^{m-1,m+1/2}(X, L)$ (as opposed to $I_{s\Phi}^{m-1,m-1/2}(X, L) + I_{s\Phi}^{\infty,m+1/2}(X, L)$). This better vanishing property of intersecting Legendre distributions makes them more useful for understanding principal type propagation.

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